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Positive Knots From Discrete Dynamical Systems Via Symbolic Dynamics

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A procedure to construct positive knots motivated by symbolic dynamics is given. It is proved that the corresponding knots have a special type of positive braids, positive permutation braids. It is proved that the constructed knots are invariant under topological conjugacy, up to period five, hence they can be used to classify discrete dynamical systems. An example is given to show that topological conjugacy failed to be an invariant for closed orbits of period more than five.

KEY WORDS: positive knots; discrete dynamics; symbolic dynamics.

1. INTRODUCTION

Symbolic dynamics (Hao and Zheng, 1998) is a coarse grained description for periodic and chaotic dynamical systems. In this formulation the exact location of the point is not important. Only its position relative to the critical point C (i.e. the point at which $f'(x) = 0$). So each point in a trajectory is given a label L or R according to whether it lies to the left or to the right of C. The trajectory is then replaced by a sequence of L's and R's which can be replaced by zeros and ones. The space of such sequences Σ can be made into a metric by the distance

$$d(s,t) = \sum_{i=0} \frac{1}{2^i} |s_i - t_i|; s = (s_0, s_1, \ldots,), t = (t_0, t_1, \ldots), s_i, t_i \in \{0, 1\}$$

On this metric space we can define a "symbolic dynamic" by the shift map

$$\sigma^k(s) = (s_k, s_{k+1}, \ldots)$$

In fact the three-dimensional flow of the nonlinear differential equation

$$\dot{x} = f(x(t))$$

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forms a map f_λ on the interval $I = [C_0, C_N]$ in the poincaré section, with $N-1$ parameters $(\lambda_1, \lambda_2, \dots, \lambda_{N-1}) \equiv \lambda$. For each point $x \in I$, there must be a unique symbolic address $A(x) \in \Sigma_N$ according to the symbol of the set to which x belongs, $x \in A(x)$. The $N-1$ critical points and the N subintervals are numbered in their natural order in symbolic space, $I_1 < C_1 < I_2 < C_2 < \cdots < I_{N-1} < C_{N-1} < I_N$. For an arbitrary admissible $N-1$ multiple sequence $W = C_1 X_1 C_2 X_2 \dots C_{N-1} X_{N-1} = \omega_1 \omega_2 \dots \omega_{|W|}$, where $|W|$ is the length of the sequence W, shifts $\varphi^i(W) = (W_{i+1}) = \omega_{i+1}\omega i + 2 \dots \omega_{|W}$, $i = 1, 2, \dots, |W|$ form $|W|$ vertices.

On the set $C^0 = \{\varphi^i(W)\}$ of vertices, a basic shift matrix is constructed by the natural shift order $\omega((W)_i, (W)_{i+1}) = 1$, which is a simple periodic matrix. The vertex set C^0 can be ordered by a permutation $\pi : Z_+ \to Z_+$ which makes $\varphi^{\pi(1)}(W) < \varphi^{\pi(2)}(W) < \cdots < \varphi^{\pi(|W|)}(W)$ hold (Hao and Zheng, 1998). This construction will be related to our work in terms of permutations in symmetric groups.

Also the functions f, σ are conjugate dynamical systems and so we can study f by analyzing σ. There is a countable infinity of periodic orbits of arbitrary period. The following convection (Hao and Zheng, 1998) is used, to order the trajectories: (1) $R > C > L$. (2) To order the two sequences Sa, Sb with a common part S, count the number of R in S. If it is even then order Sa, Sb according to the order of a, b. If the number is odd then order Sa, Sb opposite to the order of a, b. Notice that the map is decreasing on the R region. This structure can be generalized to multi-modal maps and to higher dimensional systems.

Here we introduce a procedure to construct positive knots motivated by symbolic dynamics associated to discrete dynamical systems. The constructed set is shown to contain any positive knot corresponding to a periodic orbit of a discrete dynamical system (Holmgren, 1996). It is proved that the constructed knots are invariant under topological conjugation, up to period five, hence they can be used to classify the associated discrete dynamical systems. An example is given to show that topological conjugation failed to be an invariant for closed orbits of period more than five. In fact knot theory and braid theory (Kauffman, 1996; Ahmed *et al.*, 1991) are interesting topics both mathematically, physically and biology. Knots have been related to continuous dynamical systems (Ghrist *et al.*, 1997; Elrifai and Ahmed, 1995) and discrete dynamical systems (Ghrist *et al.*, 1997; Ahmed and Elrifai, 2001).

2. PRELIMINARIES

A knot is an embedding $K : S^1 \to R^3$ on 1-sphere into 3-space. An arrow along the knot diagram can orient knots, where positive and negative crossings can be represented as in Fig. 1a, where Fig. 1b shows some kind of knots.

Braids (Birman, 1974) are those of Artin's braid group B_n which can be written as a word in powers of usual generators $\sigma_1, \sigma_2, \dots, \sigma_{n-1}$ with geometric

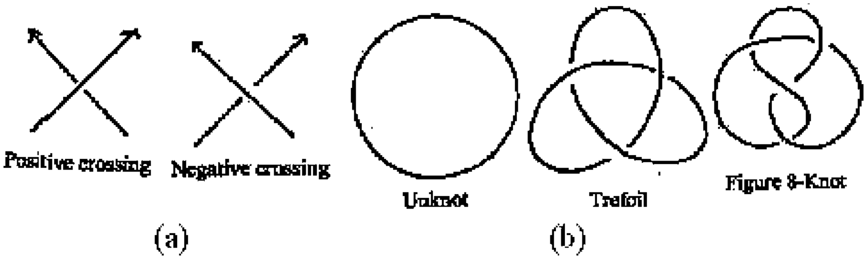


Fig. 1.

presentation as in Fig. 2a. The elements of B_n can be regarded geometrically by an arrangement of n-string running monotonically from top to bottom between two parallel discs. The generators σ_i and σ_i^{-1} represented as in Fig. 1a. One of the most common relationship between knot theory and braid groups is the closure. That is by connecting to top and bottom of each strand of the braid α to have $\hat{\alpha}$, as in Fig. 2b.

A positive braid word in B_n, is an explicitly written word in $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$. A braid is called a positive permutation braid, ppb, if it can be drawn as a geometric braid in which every pair of strings crosses at most once (Elrifai, 1988), write S_n^+ for the set of positive permutation braids. If the braids α, $\beta \epsilon S_n^+$ induce the same permutations on their strings, then $\alpha = \beta$, and for each $\pi \epsilon S_n$, the symmetric group, there is a braid $\alpha_\pi \epsilon S_n^+$ which induces that permutation (Fig. 3).

3. RESULTS

Algorithm 1. *The procedure to construct positive knots is as follows: Draw a number of points* 1, 2, ..., n *equal to the period of the orbit. Then join the points*

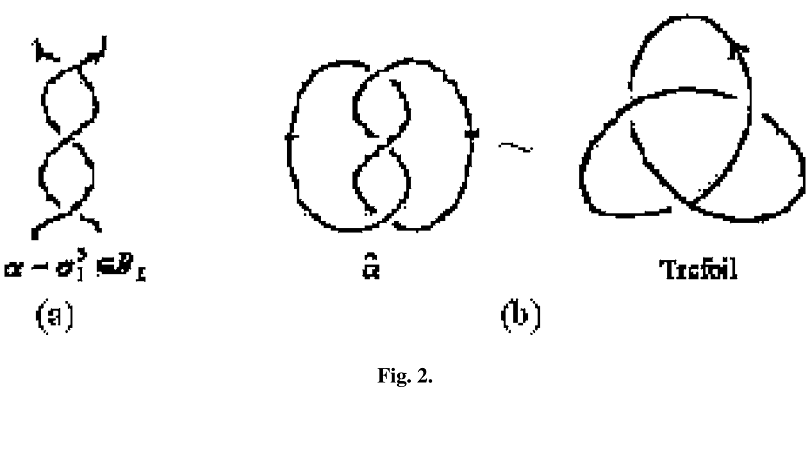


Fig. 2.

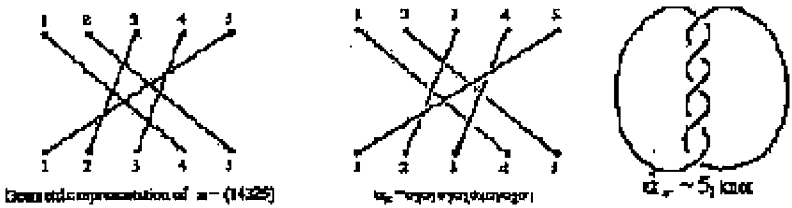


Fig. 3.

in anyway to simulate the trajectory of the periodic orbit. This means that each point is visited only once. Then as a braid no arcs cross more than once, hence it is a ppb, and no loops are allowed, which means that only the 1-cycle permutation braids are allowed. Find the positive element corresponding to such a trajectory, then find the associated knot as closed braid. Take $S^{+}_{n,1}$ for the set of all 1-cycle positive permutation braids of order n. So if the period of the orbit is n, we have $(n-1)!$ closed orbits. Hence we have the following results.

Corollary 1. *K is a knot motivated by symbolic dynamics if and only if $K = \hat{\alpha}$, for some $\alpha \epsilon S^{+}_{n,1}$ and for some positive integer n.*

Now, according to the ordering rule by distributing the sequence $W = \omega_1\omega 2 \ldots \omega_{|W|}$ with natural order $1, 2, \ldots, p-1, p$ and replacing shifted sequence with corresponding natural order, we can obtain the dynamical permutation rule $i \to \sigma(i)$, where $\sigma(i)$ denotes the natural order of sequence which is obtained by shift operation on initial sequence corresponding to natural order, which relates the present construction with the work of Hao and Zheng (1998). Now consider the following discrete dynamical system, where $g * h(t) = g(h(t))$,

$$x(t+1) = [1 - ax(t)] * x(t), \quad t = 1, 2, \ldots$$

Algorithm 2. *The periodic orbit is given by the symbolic sequence RLRC, following the trajectory they found that it follows according to the rule 1234. That the point 1 is mapped onto the point 2, which is mapped onto point 3. While the point 3 is mapped onto the point 4, where point 1 is RLRC, point 2 is LRCR, point 3 is RCRL and point 4 is CRLR. Using the ordering rule given before, it is direct to see that $1 > 3 > 4 > 2$. So the braid element, as in Fig. 4, represents the trefoil knot. Notice that in their notation RLRC means RLRL RLRL RLRL...*

Proposition 1. *The periodic orbit $(RL^n)^\infty, n \geq 1$ correspond to the unknot.*

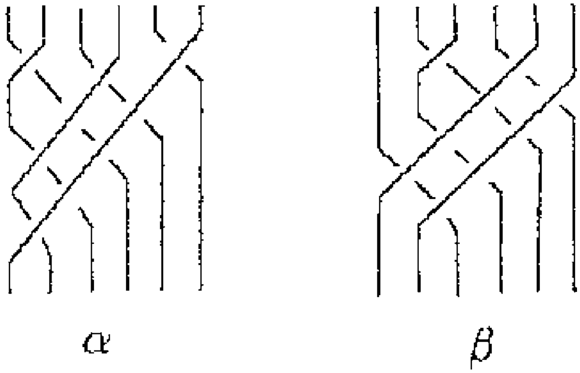


Fig. 4.

Proof: In the present constructions this orbit corresponds to the transition 123, ..., n. The corresponding braid element is $\sigma_{n-1}\sigma_{n-2}, \ldots, \sigma_2\sigma_1$, whose closure is the unknot, that is the knot which ambient isotopic to the circle. □

In general, knots constructed according to the given procedure are not invariant under topological conjugation. In Elrifai and Benkhalifa (2004) introduced a complete matrix invariant for all conjugation classes in S_n^+, $n \leq 5$, without restrictions on the cycle type of the permutation. Which implies that all periodic orbits in our procedure of period n, $n \leq 5$ are invariant under topological conjugation. Also in Morton and Hadji (2004) proved that there is only one conjugation class in $S_{n,1}^+$ for the unknotted periodic orbits, as well as all positive permutation braids in $S_{n,1}^+$ which close to the trefoil knot are conjugate, for any integer n.

4. CALCULATIONS

The only possible knots which arise, as a closed braids, from $S_{n,1}^+$, for $n < 6$, are unknot, trefoil and figure eight knots. By a direct calculations we can find that:

(1) When $n = 3$, the only closed orbit is the unknot with permutations (123), (132) and corresponding ppb representations $\sigma_2\sigma_1$, $\sigma_1\sigma_2$, respectively.
(2) When $n = 4$, the unknot and trefoil are the possible closed orbits, where the unknot comes from the permutations (1234), (1243), (1342), (1432) with corresponding ppb representations $\sigma_3\sigma_2\sigma_1$, $\sigma_2\sigma_1\sigma_3$, $\sigma_1\sigma_3\sigma_2$, $\sigma_1\sigma_2\sigma_3$, respectively. While the trefoil comes from the permutations (1324), (1423) with corresponding ppb representations $\sigma_2\sigma_1\sigma_3\sigma_2\sigma_1$, $\sigma_1\sigma_2\sigma_1\sigma_3\sigma_2$, respectively.

(3) When $n = 5$, the unknot, trefoil and figure eight knots are the possible closed orbits, where the unknot comes from (12345), (12354), (12453), (12543), (13452), (13542), (14532), (15432) the permutations. While the trefoil comes from the permutations (12453), (12534), (13245), (13254), (13524), (14253), (14352), (14523), (15342), (15423). But figure eight knot arises from (15324), (15243), (15234), (14325), (14235), (13425).

(4) The two ppbs associated to the permutations (152643), (165324) $\in S^+_{6,1}$ close to the same knot type as (2, 5) torus knot. So they represent the same closed orbit. But their associated braids

$$\alpha = \sigma_1\sigma_3\sigma_2\sigma_1\sigma_5\sigma_4\sigma_3\sigma_2\sigma_1 \quad \text{and} \quad \beta = \sigma_2\sigma_4\sigma_3\sigma_2\sigma_1\sigma_5\sigma_4\sigma_3\sigma_2$$

are not conjugate. Therefore α and β are not conjugate, in fact we can show that by different ways. Applying the algorithm in Elrifai and Benkhalifa (2004) on braids α^2 and β^2, we find their associated matrices are not equal:

$$A_{\alpha^2} = \begin{pmatrix} 4 & 5 \\ 5 & 4 \end{pmatrix}, \qquad A_{\beta^2} = \begin{pmatrix} 2 & 7 \\ 7 & 2 \end{pmatrix} \tag{1}$$

hence α^2 and β^2 are not conjugate, so that α and β are never be conjugate. Let us consider the closures of α^2 and β^2 as in Fig. 4, both are links of two components, where a_{ij} is the linking number between the two components and a_{ii} is the self crossing of each component.

5. APPLICATION

The Lorenz model consists of the equations

$$\begin{aligned} \dot{x} &= \sigma(y - x) \\ \dot{y} &= rx - y - xz \\ \dot{z} &= xy - bx \end{aligned}$$

contents three parameters r, σ and b, representing respectively the Rayleigh number, the Prandtl number and a geometric ratio. Consider the system in a wide r range at fixed $\sigma = 10$ and $b = 8/3$. The closed periodic orbits of the Lorenz system have been studied by many authors (Birman and Williams, 1983; Elrifai, 1988, 1999). It is known that, for $0 < r < 1$ the origin (0, 0, 0) is a globally stable fixed point. It loses stability at $r = 1$. A one-dimensional unstable manifold and a two-dimensional stable manifold come out from the unstable origin. The intersection of the two-dimension manifold with the Poincaré section will determine a demarcation line in the partition of the two-dimension phase plane of the Poincaré map. For $r > 1$ there appears a pair of fixed points

$$C_{\pm} = \left(\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1\right).$$

These two fixed points remain stable until r reaches 24.74. Now to see Lorenz system from the viewpoint of symbolic dynamics, we refer to Guckenheimer and Williams (1979) where they introduced the geometric Lorenz model for the vicinity of $r = 28$, which leads to symbolic dynamics of two letters.

Let us take $a \in I$ and define the finite or infinite sequence

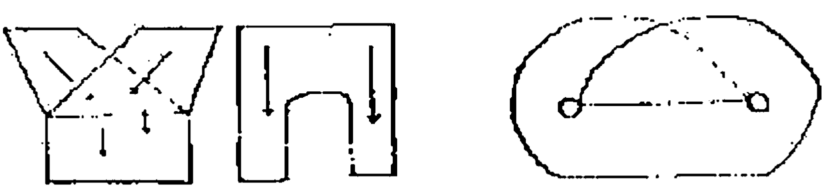
$$k(a) = k_1(a), k_2(a), k_3(a), \ldots$$

where

$$k(a) = \begin{cases} x \text{ if } a \text{ is to the left of } m \\ 0 \text{ if } a = m \\ y \text{ if } a \text{ is to the right of } m \end{cases} \quad \text{and} \quad k_i(a) = \begin{cases} x \text{ if } f^i(a) < m \\ 0 \text{ if } f^i(a) = m \\ y \text{ if } f^i(a) > m \end{cases}$$

Then sequence k are lexicographically ordered by $x < 0 < y$. We refer to Birman and Williams (1983) for the fact that the map $a \to k(a)$ is a 1-to-1 order preserving correspondence between the point of the branch set and the lexicographically ordering of the set of all sequence $k_0, k_1, \ldots$ such that each $k_i = x, y$ or 0 and the sequence terminates with $k_i = 0$. Hence the periodic orbits of the flow φ_t, which arise when the points of R^3 move simultaneously along trajectories according to the time t, correspond 1-to-1 with the cyclic permutation classes of finite aperiodic words in the free monoid generated by x and y.

Definition 1. The Lorenz knot holder is a branched two-manifold H with a boundary in S^3, consisting of one joining and one splitting chart put together, as in Fig. 5, by sewing each bottom to exactly one top and vice versa. In the joining chart, the lines come together along the branch line, where the lines leave the splitting chart at the bottom.



Lorenz attractor and Lorenz Knot holder

Fig. 5.

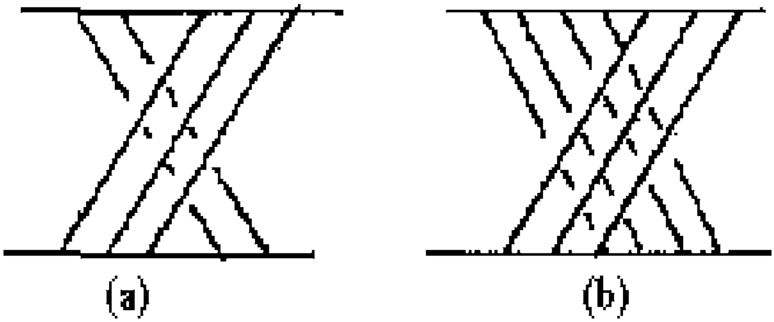


Fig. 6.

Example 1. According to the study above and the rule $x < y$, the closed orbits in knot holders in Fig. 6a and b with aperiodic words $(xy)^2x$ and $x(yx)^3$ respectively. Also in point of view of the given algorithm. This also implies that for the word $(xy)^2x$ we can write $1 \to xyxyx, 2 \to yx \ldots xyx \ldots, 3 \to yxyx \ldots x \ldots, 4 \to x \ldots xyxy \ldots, 5 \to xyx \ldots xy \ldots$, while the word $x(yx)^3$ has $1 \to xyxyxyx \ldots, 2 \to yx \ldots xyxyx \ldots, 3 \to yxyx \ldots xyx \ldots, 4 \to yxyxyx \ldots x \ldots$, $5 \to x \ldots xyx$ $yxy \ldots$, $6 \to xyx \ldots xyxy \ldots, 7 \to xyxyx \ldots xy \ldots$. The corresponding closed orbits have the trefoil 3_1 and 5_1 knot types, that according to the tabulation in Dale (1977), respectively.

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